Permutation-filtered kG-modules

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Outline of this talk

1. Some friendly objects
2. My (math) problem
   - Description of problem
   - Applications
3. The case $G = C_p$
   - grspl-projective-injectives
   - Overall strategy
   - The top region $\operatorname{supp}(\operatorname{cone} \beta_{\mathbb{E}_{\mathrm{rad}}})$
   - The right region $\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$
   - The mysterious bottom region $\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})/\langle \operatorname{cone} \beta_{\mathbb{E}_{\mathrm{rad}}} \rangle$
   - The bottom-left region $\operatorname{stab}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$
4. Future work

Outline of this talk

Some filtrations of kC_p

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Write $kC_p = k[Y]/(Y^p)$. A picture of this is

$$\begin{array}{c} \bullet \\ \downarrow Y \\ \bullet \\ \downarrow Y \\ \vdots \\ \downarrow Y \\ \bullet \\ \downarrow Y \\ \bullet \end{array}$$

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Putting the p dots in various weights produces various filtrations of kC_p, *e.g.* the radical filtration $\mathbb{E}_{\mathrm{rad}}$:

$$\begin{array}{rc} \text{wgt } 0 & \bullet \\ & \downarrow Y \\ \text{wgt } 1 & \bullet \\ & \downarrow Y \\ & \vdots \\ & \downarrow Y \\ \text{wgt } p-2 & \bullet \\ & \downarrow Y \\ \text{wgt } p-1 & \bullet \end{array}$$

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Write $[m] = k[Y]/(Y^m)$. A picture of this is

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Some operations on filtrations

In general, a filtered module is an ascending chain of submodules (starting with 0's):

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gap The filtration $\mathrm{gap}^w(A)$ where a gap is inserted at weight w so that $\mathrm{gr}^w(A) = 0$

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- $\mathrm{gr}(\mathbb{E}_{\mathrm{rad}}) = k^{\oplus p}$
- $\mathrm{un}(\mathbb{E}_{\mathrm{rad}}) = kC_p$
- $\mathbb{E}_{\mathrm{rad}}(n)$ is the filtered module

$$
\begin{array}{rc}
\text{wgt } n & \bullet \\
 & \downarrow Y \\
\text{wgt } n+1 & \bullet \\
 & \downarrow Y \\
 & \vdots \\
 & \downarrow Y \\
\text{wgt } n+p-2 & \bullet \\
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- $\mathrm{gap}^{p-1}(\mathbb{E}_{\mathrm{rad}})$ is the filtered module

$$
\begin{array}{rc}
\text{wgt } 0 & \bullet \\
 & \downarrow Y \\
\text{wgt } 1 & \bullet \\
 & \downarrow Y \\
 & \vdots \\
 & \downarrow Y \\
\text{wgt } p-3 & \bullet \\
 & \downarrow Y \\
\text{wgt } p-2 & \bullet \\
\text{wgt } p-1 & \Big\downarrow Y \\
\text{wgt } p & \bullet
\end{array}
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Some morphisms

Morphisms $f\colon A \to B$ of filtered modules respect weights: $f(A^w) \subset B^w$.

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- $\beta_A\colon A \to A(1)$ twists by 1, *e.g.*

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\begin{tikzcd}
 & \mathbb{E}_{\mathrm{rad}} \arrow[r, "\beta_{\mathbb{E}_{\mathrm{rad}}}"] & \mathbb{E}_{\mathrm{rad}}(1) \\
\text{wgt } 0 & \bullet \arrow[d, "Y"'] \arrow[dr] & \\
\text{wgt } 1 & \bullet \arrow[d, tail, "Y"'] \arrow[dr] & \bullet \arrow[d, "Y"] \\
\text{wgt } 2 & \vdots \arrow[d, "Y"'] & \bullet \arrow[d, "Y"] \\
 & \bullet \arrow[d, "Y"'] & \vdots \arrow[d, "Y"] \\
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\end{tikzcd}
$$

- $Y^a\colon \mathbb{E}_{\mathrm{rad}}(b) \to \mathbb{E}_{\mathrm{rad}}$ is a morphism iff $a \geq b$, *e.g.*

$$
\begin{tikzcd}
& \mathbb{E}_{\mathrm{rad}}(1) \arrow[r, "Y"] & \mathbb{E}_{\mathrm{rad}} \\
\text{wgt } 0 & & \bullet \arrow[d, "Y"] \\
\text{wgt } 1 & \bullet \arrow[r, "Y"] \arrow[d, "Y"'] & \bullet \arrow[d, "Y"] \\
\text{wgt } 2 & \bullet \arrow[r, "Y"] \arrow[d, "Y"'] & \bullet \arrow[d, "Y"] \\
& \vdots \arrow[d, "Y"'] & \vdots \arrow[d, "Y"] \\
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\text{wgt } p & \bullet &
\end{tikzcd}
$$

Some complexes

- Koszul complex

$$\mathrm{kos} = \left(\; k \xrightarrow{Y^{p-1}} kC_p \xrightarrow{Y} kC_p \xrightarrow{1} k \; \right)$$

Some complexes

- Koszul complex in weight 0

$$\mathrm{kos}(0) = \big(\ \mathbb{1} \xrightarrow{Y^{p-1}} kC_p(0) \xrightarrow{Y} kC_p(0) \xrightarrow{1} \mathbb{1}\ \big)$$

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- Koszul complex filtered so that gr is split exact

$$\begin{tikzcd}
\text{wgt } 0 & & & \bullet \arrow[r] \arrow[d] & \bullet \\
\text{wgt } 1 & & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
\text{wgt } 2 & & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
 & & \vdots \arrow[d] & \vdots \arrow[d] & \\
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$$\mathbb{1}(p) \xrightarrow{Y^{p-1}} \mathbb{E}_{\mathrm{rad}}(1) \xrightarrow{Y} \mathbb{E}_{\mathrm{rad}} \xrightarrow{1} \mathbb{1}$$

- Cone of a morphism (in homological degree 0), *e.g.*

$$\mathrm{cone}\, \beta_{\mathbb{E}_{\mathrm{rad}}} = \left(\mathbb{E}_{\mathrm{rad}} \xrightarrow{\beta_{\mathbb{E}_{\mathrm{rad}}}} \mathbb{E}_{\mathrm{rad}}(1) \right)$$

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The category of permutation-filtered kG-modules

Denote $\mathcal{A} = kG$-mod.
Consider the following categories and functors:

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{pfil}} \arrow[r, hook] \arrow[d, "\mathrm{gr}"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A}^{\mathrm{fil}} \arrow[d, "\mathrm{gr}"] \\
\mathrm{perm}(G;k) \arrow[r, hook] & \mathcal{A}
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Spelling it out, $\mathcal{A}^{\mathrm{pfil}}$ is the full subcategory of permutation-filtered modules. All examples we have seen so far are permutation-filtered, but *e.g.* for $G = C_p$ with p odd, the filtration

$$\begin{tikzcd}
\text{wgt } 0 & \bullet \arrow[d, "Y"'] \\
\text{wgt } 1 & \bullet \arrow[r, "Y"'] & \bullet \arrow[r, "Y"'] & \cdots \arrow[r, "Y"'] & \bullet \arrow[r, "Y"'] & \bullet
\end{tikzcd}$$

of kC_p is not.

The category of permutation-filtered kG-modules

Denote $\mathcal{A} = kG$-mod.
Consider the following tensor ($\otimes_k$ for reps) categories and tensor functors:

$$\begin{tikzcd} \mathcal{A}^{\mathsf{pfil}} \arrow[r, hook] \arrow[d, "\mathsf{gr}"'] & \mathcal{A}^{\mathsf{fil}} \arrow[d, "\mathsf{gr}"] \\ \mathrm{perm}(G;k) \arrow[r, hook] & \mathcal{A} \end{tikzcd}$$

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Consider the following exact tensor categories and exact tensor functors:

$$\begin{tikzcd}
\mathcal{A}^{\mathsf{pfil}}_{\mathsf{grspl}} \arrow[r, hook] \arrow[d, "\mathsf{gr}"'] & \mathcal{A}^{\mathsf{fil}}_{\mathsf{qab}} \arrow[d, "\mathsf{gr}"] \\
\mathsf{perm}(G; k)_{\mathsf{spl}} \arrow[r, hook] & \mathcal{A}_{\mathsf{qab}}
\end{tikzcd}$$

The category of permutation-filtered kG-modules

Denote $\mathcal{A} = kG$-mod.
Consider the following tt-categories and tt-functors:

$$\begin{tikzcd} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}) \arrow[r, hook] \arrow[d, "\mathrm{gr}"'] & \mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}}) \arrow[d, "\mathrm{gr}"] \\ \mathrm{K_b}(\mathrm{perm}(G;k)) \arrow[r, hook] & \mathrm{D_b}(\mathcal{A}) \end{tikzcd}$$

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Problem

Compute $\mathrm{Spc}(\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}))$.

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Problem

Compute $\mathrm{Spc}(\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}))$.

Recall that $\mathrm{Spc}(\mathcal{K})$ is the space of prime tt-ideals of $\mathcal{K}$ and that

$$\{\text{Thomason subsets of } \mathrm{Spc}(\mathcal{K})\}$$
$$\cong$$
$$\{\text{tt-ideals of } \mathcal{K}\}$$
$$\cong$$
$$\{\text{objects up to tt-equivalence}\}$$

The category of permutation-filtered kG-modules

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Consider the following tt-categories and tt-functors:

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Recall that $\mathrm{Spc}(\mathcal{K})$ is the space of prime tt-ideals of $\mathcal{K}$ and that

$$\begin{gathered} \{\text{closed subsets of } \mathrm{Spc}(\mathcal{K})\} \\ \wr\| \\ \{\text{tt-ideals of } \mathcal{K}\} \\ \wr\| \\ \{\text{objects up to tt-equivalence}\} \end{gathered}$$

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Representation theory Filtered representations are a natural thing to study, even with restrictions on gr.

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Motives Conjecturally $\mathrm{D_b}((kG\text{-mod})^{\mathrm{pfil}}_{\mathrm{grspl}}) \cong \mathrm{DATM}^{\mathrm{gm}}(F; k)$ are tt-equivalent, where

- F is a field containing a primitive mth root of unity
- $k = \mathbb{Z}/m$ with $\mathrm{char}(F) \nmid m$,
- $G = G_F$ is the absolute Galois group
- $\mathrm{DATM}^{\mathrm{gm}}(F; k)$ are the Artin-Tate motives, the thick triangulated (in fact rigid tt-) subcategory of $\mathrm{DM}^{\mathrm{gm}}(F; k)$ generated by $M(E)(n)$ for E/F finite separable and $n \in \mathbb{Z}$

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grspl-projective-injectives

Proposition (N., March)

$\mathcal{A}^{\text{pfil}}_{\text{grspl}}$ is Frobenius with projective-injectives forming a prime tensor-ideal $\langle \mathbb{E}_{\text{rad}} \rangle$. In fact, these are the direct sums of twists of $\mathbb{E}_{\text{rad}}$ and $kC_p(0)$.

- $\mathbb{E}_{\text{rad}}$ is projective by an elementary but intricate argument.

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- $\mathbb{E}_{\mathrm{rad}}$ is projective by an elementary but intricate argument. Given

$$\begin{tikzcd} & A \arrow[d, two heads, "p"] \\ \mathbb{E}_{\mathrm{rad}} \arrow[r, "f"'] \arrow[ur, dotted] & B, \end{tikzcd}$$

lift $f(1)$ to some $a \in A$ such that $\mathrm{wgt}\, a = 0$ and $\mathrm{wgt}\, Ya \geq 1$. A violation of weights can only occur if $w = \mathrm{wgt}\, Y^i a = \mathrm{wgt}\, Y^{i+1} a < \infty$ for some i, but then $Y^{i+1}a$ comes from a copy of $kC_p(w)$. Using this, we can correct a so that $\mathrm{wgt}\, Y^{i+1} a = \mathrm{wgt}\, Y^i a + 1$.

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- $\mathbb{E}_{\mathrm{rad}}$ is projective by an elementary but intricate argument.
- The projectives form an ideal: $\mathrm{Hom}(P \otimes A, -) = \mathrm{Hom}(P, A^\vee \otimes (-))$ is exact (the category is rigid, and every object is flat).

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- Thus the projectives are $\langle \mathbb{E}_{\mathrm{rad}} \rangle$ because $(\mathbb{E}_{\mathrm{rad}} \twoheadrightarrow \mathbb{1}) \otimes P$ exhibits P as a direct summand of $\mathbb{E}_{\mathrm{rad}} \otimes P$.

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- More elementary but intricate arguments show that projectives are direct sums of twists of $\mathbb{E}_{\mathrm{rad}}$ and $kC_p(0)$ and that $\langle \mathbb{E}_{\mathrm{rad}} \rangle$ is prime.

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- Thus the projectives are $\langle \mathbb{E}_{\mathrm{rad}} \rangle$ because $(\mathbb{E}_{\mathrm{rad}} \twoheadrightarrow \mathbb{1}) \otimes P$ exhibits P as a direct summand of $\mathbb{E}_{\mathrm{rad}} \otimes P$.
- More elementary but intricate arguments show that projectives are direct sums of twists of $\mathbb{E}_{\mathrm{rad}}$ and $kC_p(0)$ and that $\langle \mathbb{E}_{\mathrm{rad}} \rangle$ is prime.
- $\mathbb{E}_{\mathrm{rad}}^\vee = \mathbb{E}_{\mathrm{rad}}(-p+1)$ and $kC_p(0)^\vee = kC_p(0)$ and duals of projectives are injectives, so this is also the tensor-ideal of injectives.

Outline of this talk

Overall strategy for $G = C_p$

The support of cone $\beta_{\mathbb{E}_{\mathrm{rad}}}$ and the complement partition the spectrum.

$$\operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$$

$$\boxed{\operatorname{supp}(\operatorname{cone} \beta_{\mathbb{E}_{\mathrm{rad}}})}$$

$$\boxed{U(\operatorname{cone} \beta_{\mathbb{E}_{\mathrm{rad}}})}$$

Overall strategy for $G = C_p$

The tt-functor gr induces a homeomorphism onto $\operatorname{supp}(\operatorname{cone}\beta_{\mathbb{E}_{\mathrm{rad}}})$ and surjects closed points.

$$\operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}) \xleftarrow[\cong]{\operatorname{Spc}(\mathrm{gr})} \operatorname{Spc} \mathrm{K_b}(\mathrm{perm}(G;k))$$

$$\begin{tikzcd} \bullet \arrow[dr, dash] & & \bullet \arrow[dl, dash] \\ & \bullet & \end{tikzcd} \longmapsfrom \begin{tikzcd} \langle kC_p \rangle \arrow[dr, dash] & & \langle \mathrm{kos} \rangle \arrow[dl, dash] \\ & \langle kC_p, \mathrm{kos} \rangle & \end{tikzcd}$$

$$U(\operatorname{cone}\beta_{\mathbb{E}_{\mathrm{rad}}})$$

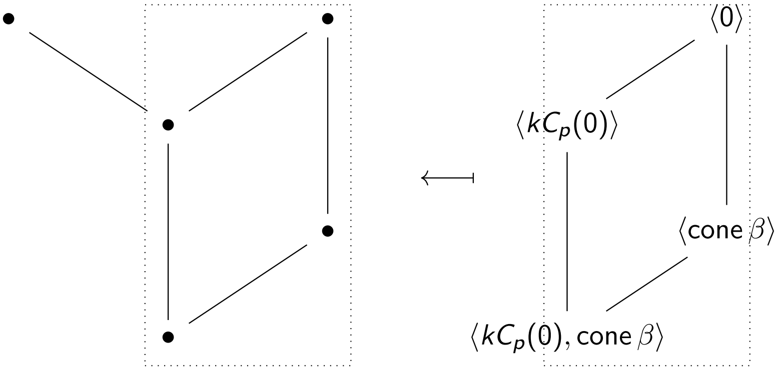
Overall strategy for $G = C_p$

Unfortunately $U(\operatorname{cone}\beta_{\mathbb{E}_{\mathrm{rad}}})$ is somewhat mysterious. But since $U(\operatorname{cone}\beta_{\mathbb{E}_{\mathrm{rad}}}) = U(\operatorname{cone}\beta) \cup U(\mathbb{E}_{\mathrm{rad}})$, we can divide and conquer.

Overall strategy for $G = C_p$

For $U(\mathrm{cone}\,\beta)$, we can do better and look at $U(\mathrm{kos}(0)) \supset U(\mathrm{cone}\,\beta)$.
Localizing at $\langle \mathrm{kos}(0) \rangle$ produces $\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$!

$$\mathrm{Spc}\,\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}) \xleftarrow{\mathrm{Spc(quo)}} \mathrm{Spc}\,\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$$



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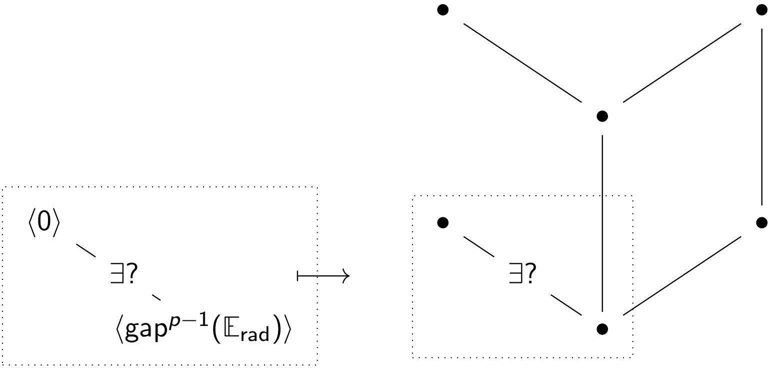
$$\operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}) \xhookleftarrow{\operatorname{Spc}(\mathrm{quo})} \operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$$

$$\begin{tikzcd}
\bullet \arrow[dr, dash] & & \langle \mathrm{kos}(0)\rangle \arrow[dl, dash] \arrow[dd, dash] & & & \langle 0 \rangle \arrow[dl, dash] \arrow[dd, dash] \\
& \langle kC_p(0), \mathrm{kos}(0)\rangle \arrow[dd, dash] & & \longmapsfrom & \langle kC_p(0)\rangle \arrow[dd, dash] & \\
& & \langle \mathrm{cone}\,\beta, \mathrm{kos}(0)\rangle \arrow[dl, dash] & & & \langle \mathrm{cone}\,\beta\rangle \arrow[dl, dash] \\
& \langle kC_p(0), \mathrm{cone}\,\beta, \mathrm{kos}(0)\rangle & & & \langle kC_p(0), \mathrm{cone}\,\beta\rangle &
\end{tikzcd}$$

Overall strategy for $G = C_p$

For $U(\mathbb{E}_{\mathrm{rad}})$, the localization is the stable category.



$$\mathrm{Spc}(\mathrm{stab}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})) \xhookrightarrow{\mathrm{Spc}(\mathrm{quo})} \mathrm{Spc}\,\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$$

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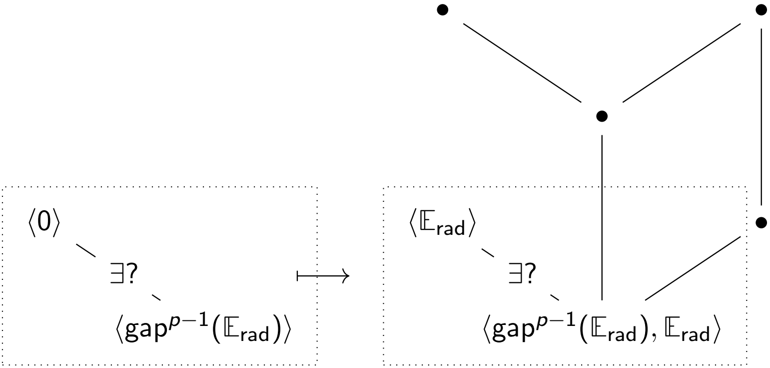

$$\mathrm{Spc}(\mathrm{stab}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})) \xhookrightarrow{\mathrm{Spc}(\mathrm{quo})} \mathrm{Spc}\,\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$$

Overall strategy for $G = C_p$

We are almost done.

$$\operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$$

$$\begin{tikzcd}
\bullet \arrow[dr, dash] & & \langle \mathrm{kos}(0) \rangle \arrow[dl, dash] \arrow[dd, dash] \\
& \langle kC_p(0), \mathrm{kos}(0) \rangle \arrow[dd, dash] & \\
\langle \mathbb{E}_{\mathrm{rad}} \rangle \arrow[dr, dashed, dash, "\exists ?"] & & \langle \mathrm{kos}(0), \operatorname{cone} \beta \rangle \arrow[dl, dash] \\
& \langle \mathrm{gap}^{p-1}(\mathbb{E}_{\mathrm{rad}}), \mathbb{E}_{\mathrm{rad}} \rangle &
\end{tikzcd}$$

Overall strategy for $G = C_p$

Simplifying gives this:

$$\mathrm{Spc}\,\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$$

$$\begin{tikzcd}
\bullet \arrow[dr, no head] & & \langle \mathrm{kos}(0)\rangle \arrow[dl, no head] \arrow[dd, no head] \\
& \langle kC_p(0), \mathrm{kos}(0)\rangle \arrow[dd, no head] & \\
\langle \mathbb{E}_{\mathrm{rad}}\rangle \arrow[dr, no head, dashed, "\exists?" description] & & \langle \mathrm{cone}\,\beta\rangle \arrow[dl, no head] \\
& \langle \mathrm{gap}^{p-1}(\mathbb{E}_{\mathrm{rad}})\rangle &
\end{tikzcd}$$

Overall strategy for $G = C_p$

Since Spc(gr) surjected the closed points, $\langle \mathbb{E}_{\mathrm{rad}} \rangle$ must specialize to something, which must be the top-left point (kos(0) is not perfect).

$$\operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$$

$$\begin{tikzcd}
\bullet \arrow[dd, dash] \arrow[dr, dash] & & \langle \mathrm{kos}(0) \rangle \arrow[dl, dash] \arrow[dd, dash] \\
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\end{tikzcd}$$

Overall strategy for $G = C_p$

Finally, the top-left point contains $kC_p(0)$, and it must be $\langle kC_p(0)\rangle$ by examining supports.

$$\operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$$

$$\begin{tikzcd}
\langle kC_p(0)\rangle \arrow[dd, dash] \arrow[dr, dash] & & \langle \mathrm{kos}(0)\rangle \arrow[dl, dash] \arrow[dd, dash] \\
& \langle kC_p(0), \mathrm{kos}(0)\rangle \arrow[dd, dash] & \\
\langle \mathbb{E}_{\mathrm{rad}}\rangle \arrow[dr, dash, "\exists?" description] & & \langle \mathrm{cone}\,\beta\rangle \arrow[dl, dash] \\
& \langle \mathrm{gap}^{p-1}(\mathbb{E}_{\mathrm{rad}})\rangle &
\end{tikzcd}$$

Overall strategy for $G = C_p$

Theorem (almost) (N.)

$\operatorname{Spc} \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})$ has the following description:

$$\begin{tikzcd}
\langle kC_p(0) \rangle \arrow[dd, dash] \arrow[dr, dash] & & \langle \mathrm{kos}(0) \rangle \arrow[dl, dash] \arrow[dd, dash] \\
& \langle kC_p(0), \mathrm{kos}(0) \rangle \arrow[dd, dash] & \\
\langle \mathbb{E}_{\mathrm{rad}} \rangle \arrow[dr, dashed, dash, "\exists ?"'] & & \langle \operatorname{cone} \beta \rangle \arrow[dl, dash] \\
& \langle \mathrm{gap}^{p-1}(\mathbb{E}_{\mathrm{rad}}) \rangle &
\end{tikzcd}$$

Outline of this talk

The top region of $\mathrm{Spc}(\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}))$

Proposition (N., March)

$\mathrm{gr}\colon \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}) \to \mathrm{K_b}(\mathrm{perm}(G;k))$ induces a homeomorphism

$$\mathrm{supp}(\mathrm{cone}\,\beta_{\mathbb{E}_{\mathrm{rad}}}) \cong \mathrm{Spc}(\mathrm{K_b}(\mathrm{perm}(G;k))) = \begin{array}{ccc} \langle kC_p \rangle & & \langle \mathrm{kos} \rangle \\ & \diagdown \quad \diagup & \\ & \langle kC_p, \mathrm{kos} \rangle & \end{array}$$

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This is the top region in the sense that $\mathrm{Spc}(\mathrm{gr})$ surjects the closed points; by ttrigonometry this is equivalent to gr being conservative.

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Some easy observations:

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- gr has a section, which by functoriality of Spc implies Spc(gr) is a homeomorphism onto its image.

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Some easy observations:

- gr has a section, which by functoriality of Spc implies Spc(gr) is a homeomorphism onto its image.
- Spc(gr) lands in $\mathrm{supp}(\mathrm{cone}\,\beta_{\mathbb{E}_{\mathrm{rad}}})$ since $\mathrm{gr}(\mathrm{cone}\,\beta_{\mathbb{E}_{\mathrm{rad}}}) = k^{\oplus p}[1] \oplus k^{\oplus p}$ is a direct sum of invertibles, so $\mathrm{cone}\,\beta_{\mathbb{E}_{\mathrm{rad}}} \notin$ any prime in $\mathrm{im}\,\mathrm{Spc}(\mathrm{gr})$

Key point

Spc(gr) surjects supp(cone $\beta_{\mathbb{E}_{\mathrm{rad}}}$). By ttrigonometry, this is equivalent to gr detecting nilpotence on cone $\beta_{\mathbb{E}_{\mathrm{rad}}}$.

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- By rigidity, reduce to the case $f\colon \mathbb{1} \to \mathbb{X}$ in $\mathrm{K_b}(\mathcal{A}^{\mathrm{pfil}})$.

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Sketch of proof, just to say 5 key words:

- By rigidity, reduce to the case $f\colon \mathbb{1} \to \mathbb{X}$ in $\mathrm{K}_{\mathrm{b}}(\mathcal{A}^{\mathrm{pfil}})$.
- Unravelling, $f^{\otimes n} \otimes \operatorname{cone} \beta_{\mathbb{E}_{\mathrm{rad}}} = 0$ is equivalent to having

$$\begin{tikzcd} & \mathbb{E}_{\mathrm{rad}} \arrow[dl, "c"'] \arrow[d, "f_0^{\otimes n}\epsilon \otimes \mathbb{E}_{\mathrm{rad}} - b\beta"] \\ (\mathbb{X}^{\otimes n})_1 \otimes \mathbb{E}_{\mathrm{rad}} \arrow[r, "d \otimes \mathbb{E}_{\mathrm{rad}}"'] & (\mathbb{X}^{\otimes n})_0 \otimes \mathbb{E}_{\mathrm{rad}} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{E}_{\mathrm{rad}}(1) \arrow[d, "b"'] \arrow[dr, "0"] & \\ (\mathbb{X}^{\otimes n})_0 \otimes \mathbb{E}_{\mathrm{rad}} \arrow[r, "d \otimes \mathbb{E}_{\mathrm{rad}}"'] & (\mathbb{X}^{\otimes n})_{-1} \otimes \mathbb{E}_{\mathrm{rad}} \end{tikzcd}$$

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Spc(gr) surjects supp(cone $\beta_{\mathbb{E}_{\mathrm{rad}}}$). By ttrigonometry, this is equivalent to gr detecting nilpotence on cone $\beta_{\mathbb{E}_{\mathrm{rad}}}$.

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Sketch of proof, just to say 5 key words:

- By rigidity, reduce to the case $f\colon \mathbb{1} \to \mathbb{X}$ in $\mathrm{K_b}(\mathcal{A}^{\mathrm{pfil}})$.
- Unravelling and using that $\mathbb{E}_{\mathrm{rad}}$ is projective, $f^{\otimes n} \otimes \mathrm{cone}\, \beta_{\mathbb{E}_{\mathrm{rad}}} = 0$ is equivalent to having

$$\begin{tikzcd} & \mathbb{E}_{\mathrm{rad}} \arrow[dl, "c"'] \arrow[d, "a - b\beta_{\mathbb{E}_{\mathrm{rad}}}"] \\ (\mathbb{X}^{\otimes n})_1 \arrow[r, "d"'] & (\mathbb{X}^{\otimes n})_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{E}_{\mathrm{rad}}(1) \arrow[d, "b"'] \arrow[dr, "0"] & \\ (\mathbb{X}^{\otimes n})_0 \arrow[r, "d"'] & (\mathbb{X}^{\otimes n})_{-1} \end{tikzcd}$$

where $a = f_0^{\otimes n}\epsilon\colon \mathbb{E}_{\mathrm{rad}} \to (\mathbb{X}^{\otimes n})_0$.

- The hypothesis $\mathrm{gr}(f) = 0$ offers a nullhomotopy $k \to \mathrm{gr}(\mathbb{X}_1)$, which trivially lifts to $kC_p(0) \to \mathbb{X}_1$. Projectivity of $\mathbb{E}_{\mathrm{rad}}$ lifts this along $\mathbb{X}_1^0 \twoheadrightarrow \mathrm{gr}^0(\mathbb{X}_1)$, producing $w\colon \mathbb{E}_{\mathrm{rad}} \to \mathbb{X}_1$.

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- Set $y = f_0\epsilon - dw \colon \mathbb{E}_{\mathrm{rad}} \to \mathbb{X}_0$, which satisfies $y(1) \in \mathbb{X}_0^1$.
- Take $b \colon \mathbb{E}_{\mathrm{rad}}(1) \to (\mathbb{X}^{\otimes n})_0$ to be the morphism such that $b(1) = y(1)^{\otimes n}$, which trivially respects weights when $n \geq p$ since $y(1)^{\otimes n} \in (\mathbb{X}^{\otimes n})_0^n$. Take $c \colon \mathbb{E}_{\mathrm{rad}} \to (\mathbb{X}^{\otimes n})_1$ to be the morphism such that

$$c(1) = \sum_{j=0}^{n} f_0\epsilon(1)^{\otimes n-j-1} \otimes w(1) \otimes y(1)^{\otimes j},$$

which respects weights since $f_0\epsilon$, w, and y are morphisms from $\mathbb{E}_{\mathrm{rad}}$.

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  which respects weights since $f_0\epsilon$, w, and y are morphisms from $\mathbb{E}_{\mathrm{rad}}$.
- Since $dw = f_0\epsilon - y$, the sum $dc(1)$ telescopes, showing that $dc = a - b\beta_{\mathbb{E}_{\mathrm{rad}}}$.

Outline of this talk

Finite permutation resolutions

Theorem (Rouquier, 2008)

$\mathrm{K_b}(\mathrm{perm}(G;k))/\langle \text{qab-ac} \rangle \to \mathrm{D_b}(\mathcal{A})$ is an equivalence

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In contrast, a kG-module admits a finite projective resolution iff it is projective.

Proposition (N., April)

$\mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})/\langle \text{qab-ac} \rangle \to \mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$ is an equivalence for G a p-group. Every filtered kG-module admits a finite resolution by permutation-filtered kG-modules.

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Key Idea

A complex $\mathbb{X}$ in $\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$ is *m-projective-permutation* if $\mathbb{X}_i$ is permutation-filtered for all i and $\mathbb{X}_j$ is qab-projective for $j \leq m$. The full subcategory of $\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$ of objects admitting an m-projective-permutation resolution for all $m \geq 0$ is triangulated.

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- We have generators for $\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$ as tt-subcategory: $\mathbb{1}(1)$, $\mathbb{1}(-1)$, and $A(0)$ for A in $\mathcal{A}$.

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- We have generators for $\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$ as tt-subcategory: $\mathbb{1}(1)$, $\mathbb{1}(-1)$, and $A(0)$ for A in $\mathcal{A}$.
- These generators admit m-projective-permutation resolutions for all $m \geq 0$, namely by taking the ones afforded by Balmer-Gallauer 2022 and putting them in a single weight. So we are done.

The right region $\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})$

Proposition (N., February)

$\mathrm{Spc}(\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}}))$ is the space

$$\begin{tikzcd}
 & & 0 \arrow[dll, dash] \arrow[dd, dash] \\
\langle kC_p(0) \rangle \arrow[dd, dash] & & \\
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\end{tikzcd}$$

This generalizes to p-groups.

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This generalizes to p-groups.

- $\mathrm{gr}\colon \mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}}) \to \mathrm{D_b}(\mathcal{A})$ detects nilpotence on $\mathrm{cone}\,\beta$, hence as before $\mathrm{Spc}(\mathrm{gr})$ is a homeomorphism onto $\mathrm{supp}(\mathrm{cone}\,\beta)$.

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This generalizes to p-groups.

- $\mathrm{gr}\colon \mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}}) \to \mathrm{D_b}(\mathcal{A})$ detects nilpotence on $\mathrm{cone}\,\beta$, hence as before $\mathrm{Spc}(\mathrm{gr})$ is a homeomorphism onto $\mathrm{supp}(\mathrm{cone}\,\beta)$.
- The bottom two points come from

$$\begin{tikzcd}
\mathrm{D_b}(\mathcal{A}) \arrow[d, "(0)"'] \\
\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}}) \arrow[d, two heads] \\
\mathrm{D_b}(\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}})/\langle \mathrm{cone}\,\beta \rangle \arrow[uu, bend right=60, "\mathrm{un}"']
\end{tikzcd}$$

being an adjoint equivalence.

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2. My (math) problem
   - Description of problem
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   - Overall strategy
   - The top region $\mathrm{supp}(\mathrm{cone}\,\beta_{\mathbb{E}_{\mathrm{rad}}})$
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A good guess

$$\begin{tikzcd} \mathrm{K_b}(\mathrm{perm}(G;k)) \arrow[d, "(0)"'] \\ \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}) \arrow[d, two heads, "\mathrm{quo}"'] \\ \mathrm{D_b}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}})/\langle \mathrm{cone}\,\beta_{\mathbb{E}_{\mathrm{rad}}}\rangle \end{tikzcd}$$

- Typically $\mathrm{quo}(0) \dashv \mathrm{un}$ are equivalences, *e.g.* for $\mathcal{A}^{\mathrm{fil}}_{\mathrm{qab}}$.

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A good guess

$\mathbb{1}(p)$ is in the essential image of quo(0) because

$$\begin{tikzcd} \operatorname{ess\,im}(\mathrm{quo}(0)) \ni \arrow[d, "\cong"] & kC_p(0) \arrow[r] \arrow[d, "\cong"] & kC_p(0) \arrow[r] \arrow[d, "\cong"] & k(0) \arrow[d, equal] \\ \mathbb{X} & \mathbb{E}_{\mathrm{rad}}(1) \arrow[r] & \mathbb{E}_{\mathrm{rad}} \arrow[r] & \mathbb{1} \\ \mathbb{1}(p) \arrow[u, "\cong"', "\text{grspl-acyclic cone}"] & \mathbb{1}(p) \arrow[u] & & \end{tikzcd}$$

We are using that mod cone $\beta_{\mathbb{E}_{\mathrm{rad}}}$, not only is $\beta_{\mathbb{E}_{\mathrm{rad}}}$ invertible, but $kC_p(0) \to \mathbb{E}_{\mathrm{rad}}$ is too.

A good guess

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- For $p = 2$, the essential image of $\mathsf{quo}(0)$ does contain $\mathbb{1}(1)$!

A good guess

When $p = 2$, we have a miraculously shorter Koszul $k \to kC_2 \to k$, so

$$\begin{tikzcd}
\operatorname{ess\,im}(\mathrm{quo}(0)) \ni \arrow[d, "\cong"] \\
\mathbb{X} \\
\mathbb{1}(p) \arrow[u, "\cong", "\text{cone is grspl-acyclic}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
kC_2(0) \arrow[r] \arrow[d, "\cong"] & k(0) \arrow[d, equal] \\
\mathbb{E}_{\mathrm{rad}} \arrow[r] & \mathbb{1} \\
\mathbb{1}(1) \arrow[u] &
\end{tikzcd}$$

shows that $\mathbb{1}(1)$ is in the essential image of $\mathrm{quo}(0)$!

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Outline of this talk

Consider the prime tt-ideal $\ker(\mathrm{un}\colon \mathrm{stab}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}) \to \mathrm{stab}(\mathcal{A}))$ of the filtrations of projectives.

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Progress (N., Sunday)

$\ker(\mathrm{un})$ is generated as a tt-ideal by any nonzero $\mathrm{gap}^i(\mathbb{E}_{\mathrm{rad}})$, and it contains any remaining primes. So $\mathrm{Spc}(\mathrm{stab}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}))$ is the space

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- For example, to show that $\mathrm{Spc}(\mathrm{stab}(\mathcal{A}^{\mathrm{pfil}}_{\mathrm{grspl}}))$ consists of only two points, it suffices to show that for any nonzero A, some nonzero $\mathrm{gap}^i(\mathbb{E}_{\mathrm{rad}})$ is in $\langle A \rangle$.

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- For $p > 2$, this does not work because there are indecomposable filtrations of decomposables.

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4. Future work

Modular fixed points and $\mathrm{K_b}(\mathrm{perm}(G; k))$

Set $\mathcal{K}(G) = \mathrm{K_b}(\mathrm{perm}(G; k))$ for G a p-group. $\mathrm{Spc}(\mathcal{K}(G))$ is known!

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Theorem (Balmer-Gallauer, 2023)

There exist modular fixed points functors $\Psi^H \colon \mathcal{K}(G) \to \mathcal{K}(G /\!\!/ H)$ satisfying $\Psi^H(k(G/K)) = k((G/K)^H)$. The cohomological opens $\mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^\bullet(G /\!\!/ H, k)) = \mathrm{Spc}(\mathrm{D_b}(G /\!\!/ H)) \subset \mathrm{Spc}(\mathcal{K}(G /\!\!/ H))$, for $H \leq G$ up to conjugacy, partition $\mathrm{Spc}\,\mathcal{K}(G)$.

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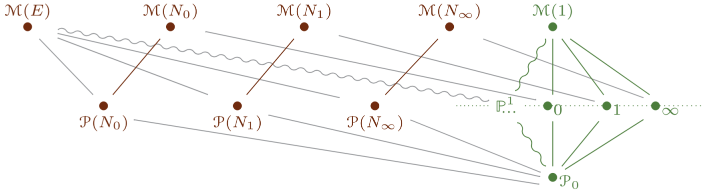
For example $\mathrm{Spc}(\mathcal{K}(C_2 \times C_2))$ is the space



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